

ON THE UNIQUENESS FOR COAGULATION AND MULTIPLE FRAGMENTATION EQUATION

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ABSTRACT. In this article, the uniqueness of weak solutions to the coagulation and multiple fragmentation equation is proved for a large range of unbounded coagulation kernels with the multiple fragmentation kernels probably having a singularity at origin. This work generalizes the preceding ones, by including some physically relevant coagulation and multiple fragmentation kernels which was not considered before.

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1. INTRODUCTION

We analyze the following continuous coagulation and multiple fragmentation equation

$$(1.1) \quad \frac{\partial f(x, t)}{\partial t} = \frac{1}{2} \int_0^x K(x-y, y) f(x-y, t) f(y, t) dy - \int_0^\infty K(x, y) f(x, t) f(y, t) dy \\ + \int_x^\infty b(x, y) S(y) f(y, t) dy - S(x) f(x, t),$$

with

$$(1.2) \quad f(x, 0) = f_0(x) \geq 0,$$

where $f(x, t)$ is the number density of particles of size $x \in \mathbb{R}_{>0} := (0, \infty)$ at time $t \geq 0$. The first two integrals on the right-hand side of (1.1) describe, respectively, the birth and death of particles of size x due to the coagulation process. The coagulation kernel $K(x, y)$ represents the rate at which particles of size x coalesce with particles of size y . The remaining two integrals, on the right-hand side of (1.1), due to the fragmentation process, can be interpreted similarly. Here the particles can fragment into more than two pieces. The breakage function $b(x, y)$ is the probability density function for the formation of particles of size x from the particles of size y . Moreover, it is assumed that b is non-negative measurable function which satisfies $b(x, y) = 0$ for $x > y$ and S is the fragmentation rate. The fragmentation rate S and breakage function b can be expressed in terms of the multiple-fragmentation kernel Γ as follows

$$(1.3) \quad S(x) = \int_0^x \frac{y}{x} \Gamma(x, y) dy, \quad b(x, y) = \Gamma(y, x) / S(y).$$

The number of fragments obtained from the breakage of particles of size y ,

$$(1.4) \quad \int_0^y b(x, y) dx = N < \infty, \quad \text{for all } y \in \mathbb{R}_{>0},$$

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and the necessary condition for mass conservation is

$$(1.5) \quad \int_0^y xb(x, y)dx = y \quad \text{for all } y \in \mathbb{R}_{>0}.$$

The typical question of existence and uniqueness of solutions to coagulation-fragmentation equation has already been discussed extensively in several articles for example [1, 3, 5, 6, 7, 9, 13, 19, 22, 23]. However, most of these articles are dedicated to the case of binary fragmentation. The case of multiple fragmentation was not considered up to that level. Best to our knowledge, the first existence and uniqueness of solutions to the continuous coagulation and multiple fragmentation equation was proved in [18] for bounded coagulation and fragmentation kernels K and F respectively. By using a different (semigroup) approach, a similar result was acquired in [17]. In the similar contest, a more relaxed result on existence and uniqueness of solutions to (1.1)-(1.2) is shown in [12] when S satisfies almost a linear growth but still with a bounded K . The case of unbounded coagulation and fragmentation kernels is later considered in [14, 8, 10, 4] where the existence of solutions to (1.1)-(1.2) is demonstrated under different growth conditions on coagulation and fragmentation kernels. In [14], the coagulation kernel K of the type $K(x, y) = r(x)r(y)$ with no growth restriction on r is assumed with a reasonable growth condition on multiple fragmentation kernel Γ . In [8], the coagulation kernel K satisfying $K(x, y) \leq \phi(x)\phi(y)$ for some sublinear function ϕ and a reasonable growth condition on Γ are assumed to show the existence of solutions. Still the fragmentation kernel Γ is required to be bounded near origin in [14, 8] which thus excludes classical fragmentation kernels discussed in the literature such as $\Gamma(y, x) = (\alpha + 2)x^\alpha y^{\gamma - (\alpha + 1)}$ with $\alpha > -2$ and $\gamma \in \mathbb{R}$, see [16]. Later, this type of fragmentation kernels are partially included in [10] which extends the previous results with the same growth conditions on the coagulation kernel K as in [8]. These type of kernels are also recently discussed in [4] but with an assumption of finiteness on higher moments. However, the uniqueness of solutions to (1.1)-(1.2) for unbounded coagulation and fragmentation kernels is only shown in [8] which does not include most of the physically relevant coagulation and fragmentation kernels considered in the existence result [8, 10].

The purpose of this work is to prove the uniqueness of solutions to (1.1) which can cover this gap partially. Let us briefly outline the manuscript. In this section, we give some hypotheses, notation of spaces, definitions and state the main result in Theorem 1.2 to demonstrate the uniqueness of solutions. A few examples of unbounded coagulation and multiple fragmentation kernels are also given at the end of the section. Finally, Theorem 1.2 is proved by showing the integrability of higher moments in Theorem 2.1 in Section 2. An inspiration to complete this work came from [23, 5, 9].

In particular, we make the following hypotheses on the kernels.

Hypotheses 1.1. (A1) K is a non-negative measurable function on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and is symmetric, i.e. $K(x, y) = K(y, x)$ for all $x, y \in \mathbb{R}_{>0}$,

(A2) $K(x, y) \leq \phi(x)\phi(y)$ for all $x, y \in \mathbb{R}_{>0}$ where $\phi(x) \leq k_1(1 + x)^\mu$ for some $0 \leq \mu \leq 1$ and constant $k_1 > 0$.

(A3) Γ is a non-negative measurable function on $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that $\Gamma(x, y) = 0$ if $0 < x < y$. Defining S and b by (1.3), we assume that b satisfies (1.4)-(1.5).

(A4) $S : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ is a measurable function. For all $x > 0$, there exist constants $m_1, m_2 > 0$ such that

$$S(x) \leq m_1(1+x)^{a_1} \quad \text{and} \quad \int_0^x (1+y)b(x,y)dy \leq m_2(1+x)^{a_2},$$

where $a_1 + a_2 \leq 1$.

(A5) There are constants $L > 0$ and $1 + \nu > 0$ such that

$$\Gamma(x, y) = b(y, x) \cdot S(x) \geq R(x) := L(1+x)^\nu \quad \text{for any } x \geq 1 \text{ and } y \in (0, x).$$

Let X be the following Banach space with norm $\|\cdot\|$

$$X = \{f \in L^1]0, \infty[: \|f\| < \infty\} \quad \text{where} \quad \|f\| = \int_0^\infty (1+x)|f(x)|dx$$

and set

$$X^+ = \{f \in X : f \geq 0 \text{ a.e.}\}.$$

The main result of this work is the following uniqueness result:

Theorem 1.2. *Let f be a solution of equation (1.1)-(1.2) with initial data $f_0 \in X^+$. If (A1), (A2), (A3), (A4), (A5) and $1 + \nu > \mu$ hold, then the solution is unique.*

The r th moment of the number density distribution f if it exists is defined by

$$M_r(t) = M_r(f(t)) := \int_0^\infty x^r f(x, t) dx, \quad r \geq 0.$$

The first two moments represent some important properties of the distribution. The zeroth ($r = 0$) and first ($r = 1$) moments are proportional to the total number and the total mass of particles, respectively.

Definition 1.3. *Let $T \in]0, \infty]$. A solution f of (1.1)-(1.2) is a non-negative function $f : [0, T[\rightarrow X^+$ such that, for a.e. $x \in]0, \infty[$ and all $t \in [0, T[$,*

(i) $s \mapsto f(x, s)$ is continuous on $[0, T[$,

(ii) the following integrals are finite

$$\int_0^t \int_0^\infty K(x, y) f(y, s) dy ds < \infty \quad \text{and} \quad \int_0^t \int_x^\infty b(x, y) S(y) f(y, s) dy ds < \infty,$$

(iii) the function f satisfies the following weak formulation of (1.1)-(1.2)

$$\begin{aligned} f(x, t) = f_0(x) + \int_0^t \left\{ \frac{1}{2} \int_0^x K(x-y, y) f(x-y, s) f(y, s) dy \right. \\ \left. - \int_0^\infty K(x, y) f(x, s) f(y, s) dy + \int_x^\infty b(x, y) S(y) f(y, s) dy - S(x) f(x, s) \right\} ds. \end{aligned}$$

Let us mention the following coagulation kernels discussed in [4, 8, 10] to show the existence of solutions to (1.1)-(1.2). We should point out these kernel will satisfy the hypotheses (A1) and (A2).

(1) Shear kernel (non-linear velocity profile), see [2, 21]

$$K(x, y) = k_0(x^{1/3} + y^{1/3})^{7/3}.$$

(2) The modified Smoluchowski kernel, see [11],

$$K(x, y) = k_0 \frac{(x^{1/3} + y^{1/3})^2}{x^{1/3} \cdot y^{1/3} + c}, \quad c > 0.$$

The third example of coagulation kernel mentioned in [8] is bounded at infinity and therefore is already covered in the previous work. It is also clear that the coagulation kernels satisfying $K(x, y) \leq x^{\mu_1} y^{\mu_2} + x^{\mu_2} y^{\mu_1}$ for some $\mu_1, \mu_2 \in [0, 1[$ which are usually used in the mathematical literature satisfy (A1)-(A2).

Remark 1.4. The classical shear kernel (with linear velocity profile), see [2], $K(x, y) = k_0(x^{1/3} + y^{1/3})^3$, and the multiplicative coagulation kernel i.e. $K(x, y) = xy$ also satisfy (A1)-(A2) and thus will also be included in this uniqueness result. However, the existence of solutions to (1.1) is still an open problem for these cases.

Now let us take the following example of multiple fragmentation kernels partially considered in [10].

$$(1.6) \quad S(y) = y^\gamma \quad \text{and} \quad b(x, y) = \frac{\alpha + 2}{y} \left(\frac{x}{y} \right)^\alpha \quad \text{for } 0 < x < y,$$

where $\gamma \in \mathbb{R}$ and $\alpha > -2$, see [16, 20]. Since this breakage function has a physical meaning only if $-2 < \alpha \leq 0$. At $\alpha = 0$, this gives us the case of binary fragmentation. Assume that $0 < \gamma \leq 1$ and $-1 < \alpha \leq 0$. Then, we have

$$\Gamma(y, x) = (\alpha + 2)x^\alpha y^{\gamma - (\alpha + 1)} \quad \text{for } 0 < x < y.$$

First, to check (A5), we interchange the roles of x and y to obtain

$$\begin{aligned} \Gamma(x, y) &= (\alpha + 2)y^\alpha x^{\gamma - (\alpha + 1)} \quad \text{for } 0 < y < x \\ &= (\alpha + 2)x^{\gamma - 1} \left(\frac{y}{x} \right)^\alpha \\ &\geq (\alpha + 2)(1 + x)^{\gamma - 1} \\ &=: L(1 + x)^\nu \quad \text{where } \nu = \gamma - 1 > -1 \quad \text{and } L = \alpha + 2. \end{aligned}$$

This shows that (A5) is satisfied. Now let us check (A4). The first part for S of (A4) is obvious. Next, we estimate

$$\begin{aligned} \int_0^x (1 + y)b(y, x)dy &= \frac{\alpha + 2}{x^{\alpha + 1}} \int_0^x (1 + y)y^\alpha dy \\ &= \frac{\alpha + 2}{x^{\alpha + 1}} \left[\frac{x^{\alpha + 1}}{\alpha + 1} + \frac{x^{\alpha + 2}}{\alpha + 2} \right] \\ &= \frac{\alpha + 2}{\alpha + 1} \left[1 + \frac{\alpha + 1}{\alpha + 2} x \right] \\ &\leq \frac{\alpha + 2}{\alpha + 1} (1 + x)^{\frac{\alpha + 1}{\alpha + 2}} \\ &\leq m_2(1 + x)^{a_2} \quad \text{where } a_2 = \frac{\alpha + 1}{\alpha + 2} \quad \text{and } m_2 = \frac{1}{a_2}. \end{aligned}$$

2. UNIQUENESS

To prove the Theorem 1.2, we need to prove the following theorem motivated by [9].

Theorem 2.1. *Let us assume that (A1), (A2), (A3), (A5), and $1 + \nu > \mu$ hold. Instead of (A4), we assume that a more general growth condition on S*

(A4') For all $x > 0$, there exists $m > 0$ such that

$$S(x) \leq m_1(1 + x)^\gamma \quad \text{for some } 0 \leq \gamma \leq \mu,$$

hold. Let $f \in X^+$ be any solution of equation (1.1)-(1.2) on $[0, T[$, $T > 0$. Then, for every $t \in [0, T[$ and for every $\delta > 0$, $I_{2+\nu-\delta}(t) < \infty$, where $I_r(t) := \int_0^t M_r(f(s))ds$.

Theorem 2.1 can be proved by applying a repeated application of the following Lemma:

Lemma 2.2. Assume (A1), (A2), (A3), (A4'), (A5) and $1 + \nu > \mu$ hold. Let $f \in X^+$ be any solution of equation (1.1) on $[0, T[$, $T > 0$, and assume $I_\rho(t) < \infty$ for all $t \in [0, T[$ and some $\rho \geq 1$ with $\rho > \mu$. Then, for all $t \in [0, T[$, $I_{\rho+\nu-\mu+1}(t) < \infty$ if $\rho - \mu < 1$. In case $\rho - \mu \geq 1$ we obtain $I_{2+\nu-\delta}(t) < \infty$ for any $\delta > 0$.

Proof. Let us begin with the Definition 1.3 (iii), multiply by x^λ , $\lambda \in [0, 1)$ on the both sides and then integrate with respect to x from 0 to n to get

$$\begin{aligned} & \int_0^n x^\lambda [f(x, t) - f_0(x)] dx \\ &= \int_0^t \left[\frac{1}{2} \int_0^n \int_0^x x^\lambda K(x-y, y) f(x-y, s) f(y, s) dy dx \right. \\ & \quad - \int_0^n \int_0^\infty x^\lambda K(x, y) f(x, s) f(y, s) dy dx \\ & \quad \left. + \int_0^n \int_x^\infty x^\lambda b(x, y) S(y) f(y, s) dy dx - \int_0^n x^\lambda S(x) f(x, s) dx \right] ds. \end{aligned}$$

Changing the order of intergrations in the first term on the right-hand side, and then substituting $x-y = x'$, $y = y'$, we obtain

$$\begin{aligned} & \int_0^n x^\lambda [f(x, t) - f_0(x)] dx \\ &= \int_0^t \left[\frac{1}{2} \int_0^n \int_0^{n-y} (x+y)^\lambda K(x, y) f(x, s) f(y, s) dx dy \right. \\ & \quad - \frac{1}{2} \int_0^n \int_0^{n-x} (x^\lambda + y^\lambda) K(x, y) f(x, s) f(y, s) dy dx \\ & \quad - \int_0^n \int_{n-x}^\infty x^\lambda K(x, y) f(x, s) f(y, s) dy dx \\ & \quad + \int_0^n \int_x^n x^\lambda b(x, y) S(y) f(y, s) dy dx \\ & \quad \left. + \int_0^n \int_n^\infty x^\lambda b(x, y) S(y) f(y, s) dy dx - \int_0^n x^\lambda S(x) f(x, s) dx \right] ds. \end{aligned}$$

Changing the order of integrations in the fourth term on the right-hand side and interchanging the roles x and y in the first and fourth terms on the right-hand side, we estimate

$$\begin{aligned} & \int_0^n x^\lambda [f(x, t) - f_0(x)] dx \\ & \quad + \int_0^t \left[\frac{1}{2} \int_0^n \int_0^{n-x} \{x^\lambda + y^\lambda - (x+y)^\lambda\} K(x, y) f(x, s) f(y, s) dy dx \right. \\ & \quad \left. + \int_0^n \int_{n-x}^\infty x^\lambda K(x, y) f(x, s) f(y, s) dy dx \right] \\ &= \int_0^t \left[\int_0^n \int_0^x y^\lambda b(y, x) S(x) f(x, s) dy dx \right. \\ & \quad \left. + \int_0^n \int_n^\infty x^\lambda b(x, y) S(y) f(y, s) dy dx - \int_0^n x^\lambda S(x) f(x, s) dx \right] ds. \end{aligned} \tag{2.1}$$

Since $0 \leq \lambda < 1$ and $f \in X^+$, the first term on the left-hand side in (2.1) is bounded independently of n and is convergent as $n \rightarrow \infty$.

Let us now estimate the last term on the left-hand side in (2.1) as

$$\begin{aligned}
& \int_0^t \int_0^n \int_0^n x^\lambda K(x, y) f(x, s) f(y, s) dy dx ds \\
& \leq k_1^2 \int_0^t \int_0^n \int_0^n x^\lambda (1+x)^\mu (1+y)^\mu f(x, s) f(y, s) dx dy ds \\
& = k_1^2 \int_0^t \int_0^n \int_0^1 x^\lambda (1+x)^\mu (1+y)^\mu f(x, s) f(y, s) dx dy ds \\
& \quad + k_1^2 \int_0^t \int_0^n \int_1^n x^\lambda (1+x)^\mu (1+y)^\mu f(x, s) f(y, s) dx dy ds \\
& = 2^\mu k_1^2 \int_0^t \int_0^n \int_0^1 x^\lambda (1+y) f(x, s) f(y, s) dx dy ds \\
& \quad + 2^\mu k_1^2 \int_0^t \int_0^n \int_1^n x^{\lambda+\mu} (1+y) f(x, s) f(y, s) dx dy ds.
\end{aligned}$$

In case $\lambda + \mu \leq \rho$ we obtain

$$\begin{aligned}
& \int_0^t \int_0^n \int_0^n x^\lambda K(x, y) f(x, s) f(y, s) dy dx ds \\
(2.2) \quad & \leq 2^\mu k_1^2 \max_{s \in [0, t]} \|f(s)\| \int_0^t [M_\lambda(f(s)) + M_\rho(f(s))] ds < \infty.
\end{aligned}$$

Now we consider the second term on the left hand side of equation (2.1) as follows

$$\begin{aligned}
& \frac{1}{2} \int_0^t \int_0^n \int_0^n \{x^\lambda + y^\lambda - (x+y)^\lambda\} K(x, y) f(x, s) f(y, s) dy dx ds \\
& \leq \frac{k_1^2}{2} \int_0^t \int_0^n \int_0^n (x^\lambda + y^\lambda) (1+x)^\mu (1+y)^\mu f(x, s) f(y, s) dy dx ds \\
& \leq k_1^2 \int_0^t \int_0^n \int_0^n x^\lambda (1+x)^\mu (1+y)^\mu f(x, s) f(y, s) dy dx ds \\
& \leq 2^\mu k_1^2 \max_{s \in [0, t]} \|f(s)\| \int_0^t [M_\lambda(f(s)) + M_\rho(f(s))] ds < \infty.
\end{aligned}$$

Hence, the LHS of equation (2.1) converges as $n \rightarrow \infty$. This implies the convergence of the RHS in equation (2.1).

In case $\lambda + \gamma \leq \lambda + \mu \leq \rho$, we have

$$\begin{aligned}
& \int_0^t \int_0^n x^\lambda S(x) f(x, s) dx ds \leq m \int_0^t \int_0^n x^\lambda (1+x)^\gamma f(x, s) dx ds \\
& \leq m \int_0^t \left[2^\mu \int_0^1 x^\lambda f(x, s) dx + 2^\mu \int_1^n x^{\lambda+\gamma} f(x, s) dx \right] ds \\
& \leq 2^\mu m \int_0^t [M_\lambda(f(s)) + M_\rho(f(s))] ds < \infty.
\end{aligned}$$

Since the remaining terms on the RHS in (2.1) are non-negative. This implies that

$$(2.3) \quad \int_0^t \int_0^\infty \int_0^x y^\lambda b(y, x) S(x) f(x, s) dy dx ds < \infty.$$

Let us take the integral

$$\int_0^\infty \int_0^x y^\lambda b(y, x) S(x) f(x, s) dy dx =: \int_0^\infty R_x f(x, s) dx$$

where

$$R_x := \int_0^x y^\lambda b(y, x) S(x) f(x, s) dy.$$

Since R_x is non-negative due to (A3). Then by using (A5), we have

$$\begin{aligned} R_x &\geq L \int_0^x y^\lambda (1+x)^\nu dy \\ &= \frac{L}{\lambda+1} (1+x)^\nu x^{1+\lambda} \\ &= \frac{L}{\lambda+1} \frac{(1+x)^{\nu+1}}{(\frac{1}{x}+1)} x^\lambda \\ (2.4) \quad &\geq \frac{L}{2(\lambda+1)} x^{\nu+\lambda+1} \text{ for any } x \geq 1. \end{aligned}$$

Substituting (2.4) for R_x and then into (2.3), we obtain

$$\frac{L}{2(\lambda+1)} \int_0^t \int_0^\infty x^{\nu+\lambda+1} f(x, s) dx ds \leq \int_0^t \int_0^\infty \int_0^x y^\lambda b(y, x) S(x) f(x, s) dy dx ds < \infty.$$

There are two cases. For $\rho - \mu < 1$, we may take maximal $\lambda = \rho - \mu$ to give $I_{\rho+\nu-\mu+1}(t) < \infty$. Otherwise, the condition $\lambda < 1$ is more restrictive, i.e. we may take $\lambda = 1 - \delta$ for any $\delta > 0$. This gives $I_{2+\nu-\delta}(t) < \infty$. \square

Proof of Theorem 2.1. Let p be the smallest positive integer satisfying

$$p(r - \mu) > \mu \text{ where } r := 1 + \nu > \mu.$$

(I). If $p > 1$. Then $p - 1$ is a positive integer and

$$0 < (p - 1)(r - \mu) < \mu.$$

Now we define $\rho_i := 1 + (i - 1)(r - \mu)$ to have

$$1 = \rho_1 < \rho_2 < \dots < \rho_{p-1} < \rho_p < 1 + \mu.$$

Applying Lemma 2.2 p times, starting with $\rho = \rho_1 = 1$, gives $I_{\rho_{p+1}}(t) < \infty$, and one more application of Lemma 2.2 with $\rho = \rho_{p+1} = 1 + p(r - \mu) > 1$ proves the result.

(II). If $p = 1$. Then we start with $\rho = 1$ and apply Lemma 2.2 two times to obtain the desired result. \square

Proof of Theorem 1.2. Let f and g be two solutions to (1.1)-(1.2) on $[0, T[$ where $T > 0$, with $f(0) = g(0)$, and set $Y = f - g$. For $n = 1, 2, 3 \dots$ we define

$$u^n(t) = \int_0^n (1+x) |Y(x, t)| dx.$$

Multiplying $|Y|$ by $(1+x)$ and applying Fubini's Theorem to Definition 1.2 (iv) above, we obtain for each n and $0 < t < T$,

$$(2.5) \quad \begin{aligned} u^n(t) = & \int_0^t \int_0^n (1+x) \operatorname{sgn}(Y(x, s)) \left[\frac{1}{2} \int_0^x K(x-y, y) \{f(x-y, s)f(y, s) - g(x-y, s)g(y, s)\} dy \right. \\ & - \int_0^\infty K(x, y) \{f(x, s)f(y, s) - g(x, s)g(y, s)\} dy \\ & \left. + \int_x^\infty b(x, y)S(y) \{f(y, s) - g(y, s)\} dy - S(x) \{f(x, s) - g(x, s)\} \right] dx ds. \end{aligned}$$

Using the substitution $x' = x - y$, $y' = y$ in the first integral on the right-hand side of (2.5) we find that

$$(2.6) \quad \begin{aligned} u^n(t) = & \int_0^t \int_0^n \int_0^{n-x} \left[\frac{1}{2} (1+x+y) \operatorname{sgn}(Y(x+y, s)) - (1+x) \operatorname{sgn}(Y(x, s)) \right] \\ & \times K(x, y) \{f(x, s)f(y, s) - g(x, s)g(y, s)\} dy dx ds \\ & - \int_0^t \int_0^n \int_{n-x}^\infty (1+x) \operatorname{sgn}(Y(x, s)) K(x, y) \{f(x, s)f(y, s) - g(x, s)g(y, s)\} dy dx ds \\ & + \int_0^t \int_0^n \int_x^\infty (1+x) \operatorname{sgn}(Y(x, s)) b(x, y) S(y) \{f(y, s) - g(y, s)\} dy dx ds \\ & - \int_0^t \int_0^n (1+x) \operatorname{sgn}(Y(x, s)) S(x) \{f(x, s) - g(x, s)\} dx ds. \end{aligned}$$

By interchanging the order of integration and interchanging the roles of x and y , the symmetry of K yields the identity

$$(2.7) \quad \begin{aligned} & \int_0^n \int_0^{n-x} (1+x) \operatorname{sgn}(Y(x, s)) K(x, y) \{f(x, s)f(y, s) - g(x, s)g(y, s)\} dy dx \\ & = \int_0^n \int_0^{n-x} (1+y) \operatorname{sgn}(Y(y, s)) K(x, y) \{f(x, s)f(y, s) - g(x, s)g(y, s)\} dy dx. \end{aligned}$$

For $x, y > 0$ and $t \in [0, T[$ we define the function r by

$$r(x, y, t) = (1+x+y) \operatorname{sgn}(Y(x+y, t)) - (1+x) \operatorname{sgn}(Y(x, t)) - (1+y) \operatorname{sgn}(Y(y, t)).$$

Using (2.7) we can show that (2.6) can be rewritten as

$$(2.8) \quad \begin{aligned} u^n(t) = & \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} r(x, y, s) K(x, y) f(x, s) Y(y, s) dy dx ds \\ & + \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} r(x, y, s) K(x, y) g(y, s) Y(x, s) dy dx ds \\ & + \int_0^t \int_0^n \int_x^\infty (1+x) \operatorname{sgn}(Y(x, s)) b(x, y) S(y) Y(y, s) dy dx ds \\ & - \int_0^t \int_0^n (1+x) \operatorname{sgn}(Y(x, s)) S(x) Y(x, s) dx ds \\ & - \int_0^t \int_0^n \int_{n-x}^\infty (1+x) \operatorname{sgn}(Y(x, s)) K(x, y) \{f(x, s) Y(y, s) + g(y, s) Y(x, s)\} dy dx ds. \end{aligned}$$

Since the fourth integral and the last term in the fifth integral on the right-hand side of (2.8) are non-negative. We may omit them. Thus we obtain, by interchanging the order of integration for the third integral,

$$\begin{aligned}
u^n(t) &\leq \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} r(x, y, s) K(x, y) f(x, s) Y(y, s) dy dx ds \\
&\quad + \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} r(x, y, s) K(x, y) g(y, s) Y(x, s) dy dx ds \\
&\quad + \int_0^t \int_0^n \int_0^y (1+x) b(x, y) S(y) |Y(y, s)| dx dy ds \\
&\quad + \int_0^t \int_0^n \int_n^\infty (1+x) b(x, y) S(y) |Y(y, s)| dy dx ds \\
&\quad - \int_0^t \int_0^n \int_{n-x}^\infty (1+x) \operatorname{sgn}(Y(x, s)) K(x, y) f(x, s) Y(y, s) dy dx ds \\
(2.9) \quad &=: \int_0^t \sum_{i=1}^5 S_i^n(s) ds.
\end{aligned}$$

Here S_i^n , for $i = 1, \dots, 5$, are the corresponding integrands in the preceding lines.

We now consider each S_i^n individually. Noting that for all $q, q_1, q_2 \in \mathbb{R}$, we have $\operatorname{sgn}(q_1)\operatorname{sgn}(q_2) = \operatorname{sgn}(q_1 q_2)$ and $|q| = q \operatorname{sgn}(q)$. We find that

$$\begin{aligned}
r(x, y, s) Y(y, s) &\leq [(1+x+y) + (1+x) - (1+y)] |Y(y, s)| \\
&\leq [(1+x) + (1+y) + (1+x) - (1+y)] |Y(y, s)| \\
(2.10) \quad &= 2(1+x) |Y(y, s)|.
\end{aligned}$$

By using (A2), Let us consider

$$\begin{aligned}
\int_0^t S_1^n(s) ds &= \frac{1}{2} \int_0^t \int_0^n \int_0^{n-x} r(x, y, s) K(x, y) f(x, s) Y(y, s) dy dx ds \\
&\leq k_1^2 \int_0^t \int_0^n \int_0^{n-x} (1+x)^{1+\mu} (1+y)^\mu f(x, s) |Y(y, s)| dy dx ds \\
&\leq k_1^2 \int_0^t \left[\int_0^1 (1+x)^{1+\mu} f(x, s) dx + \int_1^n x^{1+\mu} \left(\frac{1}{x} + 1\right)^{1+\mu} f(x, s) dx \right] u^n(s) ds \\
&= 2^{1+\mu} k_1^2 \int_0^t [M_0(f(s)) + M_{1+\mu}(f(s))] u^n(s) ds \\
&= \int_0^t \varphi_f(s) u^n(s) ds
\end{aligned}$$

where $\varphi_f(s) := 2^{1+\mu} k_1^2 [M_0(f(s)) + M_{1+\mu}(f(s))]$. Similarly, by defining $\varphi_g(s) := 2^{1+\mu} k_1^2 [M_0(g(s)) + M_{1+\mu}(g(s))]$, we estimate

$$\int_0^t S_2^n(s) ds \leq \int_0^t \varphi_g(s) u^n(s) ds.$$

Now let us consider

$$(2.11) \quad \int_0^t S_3^n(s) ds = \int_0^t \int_0^n \int_0^y (1+x) b(x, y) S(y) |Y(y, s)| dx dy ds.$$

By interchanging the roles of x and y in (2.11) and using (A4), we obtain

$$\begin{aligned} \int_0^t S_3^n(s)ds &= \int_0^t \int_0^n \int_0^x (1+y)b(y,x)S(x)|Y(x,s)|dydxs \\ &\leq m_1m_2 \int_0^t \int_0^n (1+x)^{a_1+a_2}|Y(x,s)|dxds \\ &\leq L \int_0^t u^n(s)ds, \text{ where } L = m_1m_2. \end{aligned}$$

Thus, we obtain

$$(2.12) \quad \int_0^t \left[S_1^n(s) + S_2^n(s) + S_3^n(s) \right] ds \leq \int_0^t \varphi(s)u^n(s)ds$$

where $\varphi(s) = \varphi_f(s) + \varphi_g(s) + L$ is integrable by Theorem 2.1.

Next, to solve the integral with S_4^n , we apply Fubini's theorem, then interchange the roles of x and y , and use (A4) to estimate the following integral for each $s \in [0, t]$

$$\begin{aligned} \int_0^n \int_n^\infty (1+x)b(x,y)S(y)|Y(y,s)|dydx \\ &= \int_n^\infty \int_0^n (1+y)b(y,x)S(x)|Y(x,s)|dydx \\ &\leq \int_n^\infty \int_0^x (1+y)b(y,x)S(x)[f(x,s) + g(x,s)]dydx \\ (2.13) \quad &\leq m_1m_2 \int_n^\infty (1+x)^{a_1+a_2}[f(x,s) + g(x,s)]dydx. \end{aligned}$$

The right-hand side of (2.13) is always bounded by the constant $m_1m_2 \sup_{s \in [0,t]} [\|f(s)\| + \|g(s)\|]$ and therefore the dominated convergence theorem leads to

$$(2.14) \quad \int_0^t S_4^n(s)ds \rightarrow 0 \text{ as } n \rightarrow \infty.$$

To consider the integral with S_5^n we first observe that

$$\begin{aligned} &\left| \int_0^\infty \int_0^\infty (1+x)\text{sgn}(Y(x,s))K(x,y)f(x,s)Y(y,s)dydx \right| \\ &\leq k_1^2 \int_0^\infty \int_0^\infty (1+x)^{1+\mu}(1+y)^\mu f(x,s)|Y(y,s)|dydx \\ &\leq k_1^2(M_\mu(f(s)) + M_\mu(g(s))) \left[\int_0^1 (1+x)^{1+\mu}f(x,s)dx + \int_1^\infty \left(\frac{1}{x} + 1\right)^{1+\mu}x^{1+\mu}f(x,s)dx \right] \\ &\leq 2^{1+\mu}k_1^2(M_\mu(f(s)) + M_\mu(g(s)))(M_0(f(s)) + M_{1+\mu}(g(s))) \\ &< \infty. \end{aligned}$$

Thus, by Lemma 1.2 in [9] we obtain

$$(2.15) \quad \int_0^t S_5^n(s)ds \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The sequence u^n is bounded and monotone. Thus, from (2.12), (2.14) and (2.15) we obtain

$$\begin{aligned} u(t) &:= \int_0^\infty (1+x)|Y(x,t)|dx = \lim_{n \rightarrow \infty} u^n(t) \leq \lim_{n \rightarrow \infty} \int_0^t \varphi(s)u^n(s)ds + \lim_{n \rightarrow \infty} \int_0^t [S_4^n(s) + S_5^n(s)]ds \\ &= \int_0^t \varphi(s) \int_0^\infty (1+x)|Y(x,s)|dxds, \end{aligned}$$

which can be rewritten as

$$u(t) \leq \int_0^t \varphi(s)u(s)ds$$

with $\varphi(s) \geq 0$. Then an application of Gronwall's inequality gives

$$u(t) = \int_0^\infty (1+x)|Y(x,t)|dx = 0 \quad \text{for all } t \in [0, T].$$

Therefore, we have

$$f(x, t) = g(x, t) \quad \text{for a.e. } x \in \mathbb{R}_{\geq 0}.$$

□

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